ED209

TWO DIMENSIONAL TRANSFORMATIONS

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1   TRANSFORMATION OF POINTS

Consider the results of the multiplication of a matrix $\begin{bmatrix} x & y \end{bmatrix}$ containing the coordinates of a point P and a general 2*2 transformation matrix :

$$[X][T] = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (ax + cy) & (bx + dy) \end{bmatrix} = \begin{bmatrix} x* & y* \end{bmatrix}.$$

This mathematical notation means that the initial coordinates x and y are transformed to x* and y*, where, x* = (ax + cy) and y* = (bx + dy). Implications of considering x* and y* as the transformed coordinates of the point P. Considering several special cases .

Case 1. Where a = d = 1 and c = b = 0. The transformation[T] matrix then reduces to the identity matrix.

$$[X][T] = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x* & y* \end{bmatrix}.$$

There is no change in the coordinates of the point occurs.

Case 2. d = 1, b = c = 0, i.e.

$$[X][T] = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ax & y \end{bmatrix} = \begin{bmatrix} x* & y* \end{bmatrix}$$

Since x* = ax, produces a scale change in the x component of the position vector.

Case 3. b = c = 0 i.e.

$$[X][T] = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} ax & dy \end{bmatrix} = \begin{bmatrix} x* & y* \end{bmatrix}$$

This yields a scaling of both the x and y coordinates of the original position vector P. If a is not equal to d then the scaling are not equal. If a = d >1, then a pure enlargement or scaling

of the coordinates occurs. If $0 < a = d < 1$, then a compression of the coordinates of P occurs. If a and/or d are negative, reflections through an axis or plane occur. To see this consider $b = c = 0$, $d = 1$ and $a = -1$. Then

$$[X][T] = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -x & y \end{bmatrix} = \begin{bmatrix} x* & y* \end{bmatrix}$$

And a reflection through the y axis results. If $b = c = 0$, $a = 1$ and $d = -1$, then the reflection through the x axis occurs. If $b = c = 0$, $a = d < 0$, then the reflection through the origin occurs. Note that both reflection and scaling of the coordinates involve only the diagonal terms. First consider $a = d = 1$ and $c = 0$. Thus

$$[X][T] = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & (bx + y) \end{bmatrix} = \begin{bmatrix} x* & y* \end{bmatrix}$$

Note that the x coordinates of the point P is unchanged, while y* depends linearly on the on the original coordinates. This effect is called shear. Similarly, when $a = d = 1$, $b = 0$, the transformation produces shear proportional to the y coordinates. Thus, we see that the off-diagonal terms produce a shearing effect on the coordinates of the position vector for P. Before completing our discussion of the transformation of points, consider the effect of the general 2*2 transformation when applied to the origin i.e.

$$[X][T] = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (ax + cy) & (bx + dy) \end{bmatrix} = \begin{bmatrix} x* & y* \end{bmatrix}$$

or for the origin.

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} x* & y* \end{bmatrix}$$

Here we see the origin is invariant under a general 2*2 transformation. This is a limitation which can be overcome by the use of homogeneous coordinates.

2   ROTATION

Consider the figure shown. Writing the position vectors for P and P* we have

$$P = \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} rcos\phi & rsin\phi \end{bmatrix}$$

$$P* = \begin{bmatrix} x* & y* \end{bmatrix} = \begin{bmatrix} rcos(\phi + \theta) & rsin(\phi + \theta) \end{bmatrix}$$

Transformed point has the components

$$x* = (xcos\theta - ysin\theta)$$

y* = (xsinθ + ycosθ)

In matrix form

$$[X^*] = [X][T] = \begin{bmatrix} x* & y* \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

Thus the transformation for a general about the origin by an arbitrary angle θ is

$$[T] = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

In general, transformations with a determinant identically equal to +1 yield pure rotations. Transpose of [T] is

$$[T]^T = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} = [T]^{-1}$$

3 REFLECTION

Whereas a pure two-dimensional rotation in the xy plane occurs entirely in the two-dimensional plane about an axis normal to the xy plane, a reflection is a 180 degrees rotation out into three space and back into two space about an axis in the xy plane. A reflection about y =0 , the axis, is obtained by using

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Similarly reflection about x = 0, the y-axis, is given by

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

A reflection about the line y = x occurs

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Similarly, a reflection about the line y = -x is given by

$$[T] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

4 TRANSLATION AND HOMOGENEOUS COORDI-NATES

Translating the origin or any other in the two-dimensional plane i.e.

X* = ax + cy + m

Y* = bx + dy + n

The general transformation matrix is now 3*3

$$[T] = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{bmatrix}$$

Where the element a,b,c,d of the upper left 2*2 sub-matrix have exactly the same effects revealed by our previous discussions. m,n are the translation factors in the x and y directions, respectively. The pure two-dimensional translation matrix is

$$\begin{bmatrix} x* & y* & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix} = \begin{bmatrix} (x+m) & (y+n) & 1 \end{bmatrix}$$

Notice that now every point in the two-dimensional plane, even the origin (x = y = 0), can be transformed.

5 ROTATION ABOUT AN ARBITRARY POINT

A rotation about a point arbitrary point can be accomplished by first translating the point to the origin, performing the required rotation, and then translating the result back to the original center of rotation. Thus rotation of the position vector $\begin{bmatrix} x & y & 1 \end{bmatrix}$ about the point m,n through an arbitrary angle can be accomplished by

$$\begin{bmatrix} x* & y* & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m & -n & 1 \end{bmatrix} \begin{bmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix}$$

6 PROJECTION - A GEOMETRIC INTERPRETATION OF HOMOGENEOUS COORDIANTES

In general 3*3 transformation matrix for two-dimensional homogeneous coordinates can be subdivided into four parts:

$$[T] = \begin{bmatrix} a & b & . & . & p \\ c & d & . & . & q \\ . & . & . & . & . \\ m & n & . & . & s \end{bmatrix}$$

Where a,b,c,d produce scaling, rotation, reflection and shearing and m and n produce transla-
tion. When p = q = 0 and s=1, the homogeneous coordinate of the transformed position vectors
is always h = 1. Geometrically this result is interpreted as confining the transformation to the
h = 1 physical plane. To the show the effect of p not equal to zero and q not equal to zero in
the third column in the general 3*3 transformation matrix, consider the following:

$$\begin{bmatrix} X & Y & h \end{bmatrix} = \begin{bmatrix} hx & hy & h \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix} \text{ equation}(1)$$

$$= \begin{bmatrix} x & y & (px + qy + 1) \end{bmatrix}$$

Here X = hx, Y = hy and h = px + qy + 1. The transformed position vector expressed
in homogeneous coordinates now lies in a plane in three - dimensional space defined by h= px +
qy+ 1. This transformation is shown in figure, where the line AB in the physical (h = 1)plane is
transformed to the line CD in the h not equal to one plane, i.e. pX + qY - h+ 1 = 0. However
the results of interest are those in the physical plane corresponding to h= 1.

These results can be obtained by geometrically projecting CD from h not equal to one plane
back on to the h = 1 plane using a pencil of rays through the origin. From figure, using similar
triangles x* = X/h y* = Y/h or in homogeneous coordinates

$$\begin{bmatrix} x* & y* & 1 \end{bmatrix} = \begin{bmatrix} X/h & Y/h & 1 \end{bmatrix}$$

Now normalizing equation (1) by dividing through by the homogeneous coordinates value h
yields

$$\begin{bmatrix} x* & y* & 1 \end{bmatrix} = \begin{bmatrix} X/h & Y/h & 1 \end{bmatrix} = \begin{bmatrix} x/(px + qy + 1) & y/(px + qy + 1) & 1 \end{bmatrix}$$

x* = X/h = x/(px+qy+1)
y* = Y/h = y/(px+qy+1)